AIAA JOURNAL
Vol. 35, No. 4, April 1997

Parametric Identification of Nonlinear Structural Dynamic Systems Using Time Finite Element Method

Rakesh K. Kapania* and Sungho Park†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0203

The time finite element method (TFM) is employed for parametric identification of nonlinear structural dynamic systems. An advantage of TFM is the ease with which one can calculate the sensitivity of the transient response with respect to various design parameters, a key requirement for gradient-based parameter identification schemes. The method is simple, because one obtains the sensitivities of the response to system parameters by differentiating the algebraic equations, not original differential equations. These sensitivities are used in the Levenberg–Marquardt iterative direct method to identify parameters for nonlinear single- and two-degree-of-freedom systems. The measured response was simulated by integrating the example nonlinear systems using the given values of the system parameters. The accuracy and the efficiency of the present method are compared to a previously available approach that employs a multistep method to integrate nonlinear differential equations. It is seen, for the same accuracy, that the present approach requires fewer data points.

Nomenclature

$\boldsymbol{a}$ = linear term in variational formulation
$\boldsymbol{B}$ = bilinear term in variational formulation
$\boldsymbol{p}$ = system parameters
$\boldsymbol{q}$ = generalized coordinates
T_0 = initial time
T_f = final time
$\{\ \}$ = column vector
$\langle\ \rangle$ = row vector

Introduction

THERE is considerable interest in the control of transient response of structures under external disturbances. The design of the control system is made difficult by the fact that simple analytical models that often are used in the design phase are not adequate for the controller design. Identification techniques are used to determine the system parameters to accurately determine the system model that can be used for controller design. Although a number of studies are available for the identification of linear systems, only a few such studies are available for nonlinear systems. These include methods based on the method of multiple scales, iterative and noniterative direct methods, and state-space mappings. A review of the nonlinear system identification used in the structural, mechanical, and control engineering is given by Natke et al.[1] Their focus was on the detection of nonlinearities, the formation of mathematical models, and techniques for parameter identification. Some of the identification techniques for nonlinear systems currently being used are linear realization theory, statistical linearization, and the use of extended Kalman filter.

Batill and Bacarro[2] identified the parameters in a highly nonlinear differential equation governing the motion of an aircraft landing gear. The identification process involved relating the variation of the equations in the state variables to the corresponding equations dealing with the initial conditions. The variation of the error function is then made to vanish via changes in the parameters, which are treated as state variables, along with the displacements and velocity of the system. Mook[3] developed a technique for processing noisy state-observable, discrete time-domain measurements of a nonlinear dynamic system to estimate both the state trajectory and the model error through satisfaction of a covariance constraint. Using a number of examples, he showed that the method is capable of identifying unknown model parameters based on a least-squares formulation. Normann and Kapania[4] presented a method that was based on single- and multiple-step methods of integrating nonlinear differential equations. The system parameters for a number of examples were determined using the iterative direct method. Most recently, Hamel and Jategaonkar[5] presented a review of the successful applications of various system identification techniques to identify parametric models for flight vehicles. The present aircraft parameter estimation is mainly categorized into three parts, namely, instrumentation and filters, flight test techniques, and analysis of flight data. The methods of data analysis used for the aircraft parameter estimation include the equation error method, the output error method, the filter error method, and neural-network-based methods. They demonstrated the successful application of system identification methodology to a broad range of flight-vehicle modeling problems using the selected examples.

Most of the system identification methods are based on minimization of the square of the error between the measured response and that of the identified model. This is the classic least-squares approach in which the error is minimized by treating the problem as an unconstrained optimization problem. Most of the algorithms for solving unconstrained optimization problems require sensitivity of the response with respect to various system parameters. These sensitivities are often obtained using either finite difference or by solving a large set of differential equations. In this paper, an alternative approach, based on the time finite element method (TFM), is employed to identify a series of single-degree-of-freedom and one two-degree-of-freedom nonlinear systems.

The TFM, introduced by Argyris and Scharpf,[6] discretizes time into a number of finite elements and the response history over each element is expressed in terms of basis functions in the time coordinate. The transient response can be determined by solving a set of linear algebraic equations in the case of linear systems and a set of nonlinear equations for the case of nonlinear systems. An important development in the evolution of TFM was a series of papers by Bailey[7–10] in which the author pointed out the need for applying Hamilton's law of varying action, briefly, Hamilton's law, not Hamilton's principle, in solving problems in elastodynamics. Using Hamilton's law, Bailey used the classic Ritz method, with simple polynomials as the basis functions, to study elastodynamic response of beams. In 1988, Peters and Izadpanah[11] offered a bilinear formulation of elastodynamics as an alternative source to

Presented as Paper 96-1393 at the AIAA/ASME/ASCE/AHS/ASC 37th Structures, Structural Dynamics, and Materials Conference, Salt Lake City, UT, April 15–17, 1996; received June 11, 1996; revision received Dec. 20, 1996; accepted for publication Jan. 7, 1997.

*Professor, Department of Aerospace and Ocean Engineering. Associate Fellow AIAA.
†Graduate Student, Department of Aerospace and Ocean Engineering.

develop approximate methods to solve initial value problems. Using the proposed augmented bilinear formulation and the h-, p-, and the hp-versions, they solved a number of examples related to dynamic response of linear systems. The method has emerged as a viable method to solve initial value problems.

Kapania and Park,[12] in a recent study, extended the bilinear formulation suggested by Peters and Izadpanah for linear undamped systems to linear and nonlinear damped systems. In addition to the transient response calculations, the TFM is natural for obtaining the sensitivity of the transient response of linear and nonlinear and damped and undamped systems, because the sensitivities can be obtained easily by performing direct differentiation of the algebraic equations resulting from the application of the TFM. The proposed approach for obtaining the sensitivity is simpler than that of the direct differentiation approach because, in the TFM, differentiation of the algebraic equations and not that of the original differential equations is performed. An advantage of the present method over the finite difference approach, the most common way to find the sensitivity, is that one does not need to perform a convergence study to select an appropriate step size for obtaining the sensitivities. Also, the method can be applied as a step-by-step procedure, thereby avoiding the need for dealing with large matrices.

The TFM, along with the iterative direct method,[13] is applied to a number of nonlinear single- and two-degree-of-freedom systems. At first, in parameter identification, an objective function is formulated as a quadratic functional between the measured response of the given system and the analytic response of the mathematical model. Then the minimization of the objective function is performed to determine the system parameters. Here we adopt the Levenberg[14] and Marquardt[15] method, which involves only first partial derivatives of the response with respect to various system parameters for the minimization process. The noise is added to a simulated response for studying the effects of measurement errors on the identification procedures. The numerical results are compared with those available from a previous study by Normann and Kapania[4] and Kapania and Normann.[16] Considering all of the advantages and the numerical results, it is clear that the TFM is very much suitable for system identification.

Time Finite Element Formulation

Bilinear Form for a Linear System

The time finite element discretization, applied to the bilinear form corresponding to the governing equation for a single-degree-of-freedom spring-mass-damper system, can be written as (described in Ref. 13):

$$\begin{bmatrix} \boldsymbol{B} & \{M\psi_i(T_f)\} \\ \langle \phi_j(T_0) \rangle & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{a}* \\ [u(t)]_{t=T_0} \end{Bmatrix} \tag{1}$$

where

$$B_{ij} = \int_{T_0}^{T_f} (K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j)\,\mathrm{d}t$$
$$a_i^* = M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} F\psi_i\,\mathrm{d}t \qquad \lambda = [\dot{u}(t)]_{t=T_f} \tag{2}$$

and i, j are row and column indices, respectively. Also, M, C, and K are, respectively, the mass, damping, and stiffness coefficients, and $F(t)$ is the externally applied dynamic load. In obtaining Eq. (1), the response $u(t)$ was assumed to be

$$u(t) = \sum_{j}^{N} q_j\phi_j(t) \tag{3}$$

where ϕ_j are the basis functions. In this study, Legendre polynomials (though same results also were obtained using integrated Legendre polynomials) were employed as the basis functions.

Transient Response of a Nonlinear System

The differential equation describing a nonlinear oscillatory system over a given length of time, $T_0 < t \leq T_f$, may have a general form:

$$g(u, \dot{u}, \ddot{u}, t, \boldsymbol{p}) = 0 \tag{4}$$

where g may be nonlinear functions of u and $\dot{u}$.

The time finite discretization for Eq. (4) yields a general form:

$$\tilde{g}(\boldsymbol{q}, \boldsymbol{p}) = 0 \tag{5}$$

where p_k, $k = 1, 2, \ldots, K$ are the K design parameters and the vector $\boldsymbol{q}$ denotes the generalized coordinates.

Equation (5) can be written as

$$\tilde{g} = \boldsymbol{a} - \boldsymbol{Bq} = 0 \tag{6}$$

where $\boldsymbol{a}$ is the load vector and $\boldsymbol{B}$ is the nonlinear stiffness matrix and a function of generalized coordinates $\boldsymbol{q}$. The most obvious and direct way to solve Eqs. (5) and (6) is by an iterative method. The iteration is terminated when an error, i.e.,

$$\boldsymbol{e} = \boldsymbol{q}^{(n)} - \boldsymbol{q}^{(n-1)} \tag{7}$$

becomes sufficiently small. Usually some norm of the error is determined and iteration continues until this norm is sufficiently small. The stopping criterion is to iterate until

$$\frac{\left\| \boldsymbol{e}^{(k)} - \boldsymbol{e}^{(k-1)} \right\|_\infty}{\left\| \boldsymbol{e}^{(k)} \right\|_\infty} \leq \varepsilon (= 5.0 \times 10^{-6}) \tag{8}$$

Generally the iteration process for the problems considered in this paper converged within a small number of iterations using arbitrary initial guess values.

Transient Response Sensitivity of a Nonlinear System

The sensitivity of the transient response of a nonlinear system can be obtained by taking the derivative of Eq. (5) with respect to p_k. After simplification, one obtains

$$\frac{\partial \tilde{g}_i}{\partial p_k} + \sum_{j}^{N} \frac{\partial \tilde{g}_i}{\partial q_j}\frac{\partial q_j}{\partial p_k} = 0 \tag{9}$$

Note that from Eq. (9) it is clear that the design sensitivity equation is linear even though the analysis problem is nonlinear; $\partial \tilde{g}_i / \partial q_j$ is called the Jacobian or the tangent stiffness matrix. Because the vector of generalized coordinates q_j is already available from the transient response analysis, the first derivatives of the generalized coordinates $\partial q_j / \partial p_k$ can be calculated easily by solving Eq. (9):

$$\frac{\partial q_j}{\partial p_k} = -\left[\frac{\partial \tilde{g}_i}{\partial q_j}\right]^{-1} \frac{\partial \tilde{g}_i}{\partial p_k} \tag{10}$$

In matrix form, the sensitivity equation can be obtained by taking the derivatives of Eq. (6) with respect to p_k:

$$\frac{\partial a_i}{\partial p_k} - \sum_{j}^{N} \frac{\partial B_{ij}}{\partial p_k} q_j - \sum_{m}^{N}\left(\sum_{j}^{N} B_{ij}\frac{\partial q_j}{\partial q_m} + \sum_{j}^{N}\frac{\partial B_{ij}}{\partial q_m} q_j\right)\frac{\partial q_m}{\partial p_k} = 0 \tag{11}$$

This equation may be written symbolically as

$$[B_{ij}]\left\{\frac{\partial q_j}{\partial p_k}\right\} + \left[\frac{\partial B_{ij}}{\partial p_k}\right]\{q_j\} = \left\{\frac{\partial a_i}{\partial p_k}\right\}$$
$$-\left\{\sum_{j}^{N}\left(\sum_{m}^{N}\frac{\partial B_{im}}{\partial q_j} q_m\right)\frac{\partial q_j}{\partial p_k}\right\} \tag{12}$$

This reduces to

$$[B_{ij}^*]\left\{\frac{\partial q_j}{\partial p_k}\right\} + \left[\frac{\partial B_{ij}}{\partial p_k}\right]\{q_j\} = \left\{\frac{\partial a_i}{\partial p_k}\right\} \tag{13}$$

where

$$B_{ij}^* = B_{ij} + \sum_{m}^{N}\frac{\partial B_{im}}{\partial q_j} q_m \tag{14}$$

Applications of the aforementioned equations for determining sensitivity of transient responses of a large number of linear and nonlinear problems are given elsewhere.[13]

Parameter Identification

Iterative Direct Method

As a first step in parameter identification, an objective function is formulated as a quadratic functional between the measured response of the given system and an analytical response of the mathematical model. Then the system parameters can be determined using the method presented by Levenberg and Marquardt for minimizing the objective function. To avoid large computational costs involved with obtaining second-order partial derivatives with respect to the design parameters, the method involves only first partial derivatives of the response with respect to various system parameters.

Objective-Function Formulation

The objective function for a single-degree-of-freedom system is given as

$$L(\boldsymbol{p}) = \int_0^T (u_a - u_m)^2 \,\mathrm{d}t \tag{15}$$

and the same for a two-degree-of-freedom-system is given as

$$L(\boldsymbol{p}) = \int_0^T \left\{(u_{1a} - u_{1m})^2 + (u_{2a} - u_{2m})^2\right\}\mathrm{d}t \tag{16}$$

where u_a and u_m, respectively, denote the time series of the analytical and measured displacement, velocity, or acceleration response; T is the record length of the measured response; and $\boldsymbol{p}$ is the parameter vector to be determined. The time series u_a is an analytical solution of the assumed model for the given system. The measured data u_m were simulated by using given parameters to obtain the response of the system. Effect of random noise on the identification of the parameters also is studied by corrupting the analytical solution with random noise with varying rms values. The objective function is minimized by setting the partial derivatives of L with respect to various system parameters equal to zero. The vector of first derivatives of $L(\boldsymbol{p})$ with respect to design parameter p_k, for the single-degree-of-freedom system, yields

$$\frac{\partial[L(\boldsymbol{p})]}{\partial p_k} = \int_0^T 2(u_a - u_m)\left(\frac{\partial u_a}{\partial p_k}\right)\mathrm{d}t \tag{17}$$

Similarly, the vector of first derivatives of $L(\boldsymbol{p})$ with respect to p_k, for the two-degree-of-freedom-system, can be expressed as

$$\frac{\partial[L(\boldsymbol{p})]}{\partial p_k} = \int_0^T 2\left\{(u_{1a} - u_{1m})\left(\frac{\partial u_{1a}}{\partial p_k}\right) + (u_{2a} - u_{2m})\left(\frac{\partial u_{2a}}{\partial p_k}\right)\right\}\mathrm{d}t \tag{18}$$

Hence, $\partial u_a/\partial p_k$ for all k are the first-order sensitivity of u_a with respect to kth system parameter. Note that using the TFM, these sensitivities are obtained by simply solving the set of linear algebraic equations and not by solving a large set of ordinary differential equations as was done in the previous studies. These sensitivities are obtained by solving the sensitivity equations formulated in Eq. (13). The time series for u_a and $\partial u_a/\partial p_k$ then are used to minimize the objective function $L(\boldsymbol{p})$.

Objective Function Minimization by Levenberg–Marquardt Method

Minimization of the objective function $L(\boldsymbol{p})$ is accomplished by Newton's method. If $\boldsymbol{p}^{(i)}$ denotes the trial values of $\boldsymbol{p}$ after the ith iteration, then $\boldsymbol{p}^{(i+1)}$ is obtained as

$$\boldsymbol{p}^{(i+1)} = \boldsymbol{p}^{(i)} + h^{(i)}\Delta\boldsymbol{p} \tag{19}$$

where $\Delta\boldsymbol{p}$ is the correction vector and $h^{(i)}$ is the step size that is set equal to one. To calculate the correction vector such that, at each iteration, the value of objective function $L(\boldsymbol{p})$ will decrease most rapidly, a steepest-descent-type procedure is adopted. In general, the steepest-descent direction is the negative gradient of the function with respect to the design parameters p_k and takes the form

$$\boldsymbol{g} = -\left\{\frac{\partial L}{\partial p_1}, \frac{\partial L}{\partial p_2}, \ldots, \frac{\partial L}{\partial p_k}\right\}^T \tag{20}$$

where $\partial L/\partial p_k$ is the rate of change of L with respect to the design parameters p_k. The Hessian $\boldsymbol{H}$ of the objective function is of the form

$$\boldsymbol{H} = \frac{\partial}{\partial p_k}\left(\frac{\partial L}{\partial p_l}\right) \tag{21}$$

Therefore, the equation for correction vector $\Delta\boldsymbol{p}$ takes the form

$$\boldsymbol{H}\Delta\boldsymbol{p} = -\boldsymbol{g} \tag{22}$$

At each iteration, the gradient and Hessian are calculated, and then, a new vector $\boldsymbol{p}^{(i+1)}$ is found. The iteration is terminated when a predefined convergence criterion is met. The gradient of the objective function L for the single-degree-of-freedom system takes the form

$$\boldsymbol{g} = 2\int_0^T \left(\frac{\partial u_a}{\partial p_k}\right)(u_a - u_m)\,\mathrm{d}t \tag{23}$$

and the gradient of the two-degree-of-freedom system can be written as

$$\boldsymbol{g} = 2\int_0^T \left\{\left(\frac{\partial u_{1a}}{\partial p_k}\right)(u_{1a} - u_{1m}) + \left(\frac{\partial u_{2a}}{\partial p_k}\right)(u_{2a} - u_{2m})\right\}\mathrm{d}t \tag{24}$$

Also, the Hessian for the single-degree-of-freedom system is given by

$$\boldsymbol{H} = 2\int_0^T \left[\left(\frac{\partial u_a}{\partial p_k}\right)\left(\frac{\partial u_a}{\partial p_l}\right) + \frac{\partial}{\partial p_k}\left(\frac{\partial u_a}{\partial p_l}\right)(u_a - u_m)\right]\mathrm{d}t \tag{25}$$

whereas the Hessian for the two-degree-of-freedom system is expressed as

$$\boldsymbol{H} = 2\int_0^T \left[\left(\frac{\partial u_{1a}}{\partial p_k}\right)\left(\frac{\partial u_{1a}}{\partial p_l}\right) + \frac{\partial}{\partial p_k}\left(\frac{\partial u_{1a}}{\partial p_l}\right)(u_{1a} - u_{1m}) + \left(\frac{\partial u_{2a}}{\partial p_k}\right)\left(\frac{\partial u_{2a}}{\partial p_l}\right) + \frac{\partial}{\partial p_k}\left(\frac{\partial u_{2a}}{\partial p_l}\right)(u_{2a} - u_{2m})\right]\mathrm{d}t \tag{26}$$

where k, l are row and column indices, respectively. Equations (25) and (26) require second derivatives for the calculation of the Hessian. Usually, the calculation of second derivatives requires large computational costs. To avoid these second-derivative calculations, the Levenberg–Marquardt method is used. In this method, the correction vector is obtained by solving the following set of algebraic equations. Specifically, the equation at the ith iteration has the form

$$\left[\boldsymbol{N}^{*(i)} + \lambda^{(i)}\boldsymbol{I}\right]\Delta\boldsymbol{p}^{*(i)} = \boldsymbol{g}^{*(i)} \tag{27}$$

where

$$g_k^{*(i)} = \frac{g_k^{(i)}}{\sqrt{N_{kk}^{(i)}}}, \qquad N_{kl}^{*(i)} = \frac{N_{kl}^{(i)}}{\sqrt{N_{kk}^{(i)}N_{ll}^{(i)}}}, \qquad \Delta p_k^{*(i)} = \Delta p_k\sqrt{N_{kk}^{(i)}} \tag{28}$$

for the single-degree-of-freedom system,

$$g_k^{(i)} = 2\int_0^T \left(\frac{\partial u_a^{(i)}}{\partial p_k}\right)\left(u_a^{(i)} - u_m\right)\mathrm{d}t$$

$$N_{kl}^{(i)} = 2\int_0^T \left(\frac{\partial u_a^{(i)}}{\partial p_k}\right)\left(\frac{\partial u_a^{(i)}}{\partial p_l}\right)\mathrm{d}t \tag{29}$$

and for the two-degree-of-freedom system,

$$g_k^{(i)} = 2\int_0^T \left[\left(\frac{\partial u_{1a}^{(i)}}{\partial p_k}\right)\left(u_{1a}^{(i)} - u_{1m}\right) + \left(\frac{\partial u_{2a}^{(i)}}{\partial p_k}\right)\left(u_{2a}^{(i)} - u_{2m}\right)\right]\mathrm{d}t$$

$$N_{kl}^{(i)} = 2\int_0^T \left[\left(\frac{\partial u_{1a}^{(i)}}{\partial p_k}\right)\left(\frac{\partial u_{1a}^{(i)}}{\partial p_l}\right) + \left(\frac{\partial u_{2a}^{(i)}}{\partial p_k}\right)\left(\frac{\partial u_{2a}^{(i)}}{\partial p_l}\right)\right]\mathrm{d}t \tag{30}$$

where λ is a scaling factor, chosen to increase the size of the correction vector components if the objective-function value has been found to decrease in the preceding step. The value of λ is set as 0.01 initially and changed by a factor of 10 during successive iterations according to the objective-function values. To obtain $\lambda^{(i+1)}$, $\lambda^{(i)}$ is multiplied by 10 if the objective function has increased and divided by 10 if the objective function has decreased. After the system is solved for $\Delta \boldsymbol{p}*$,

$$\Delta p_k = \frac{\Delta p_k^{*(i)}}{\sqrt{N_{kk}^{(i)}}} \tag{31}$$

each component of the scaled version of the correction vector has to be scaled back using Eq. (31).

For one of the examples studied here, the conjugate gradient method[17,18] also was applied to minimize the objective function.

Numerical Results

The performance of the proposed TFM-based approach was evaluated by identifying parameters for a number of single- and two-degree-of-freedom systems. The parameters were identified from the impulse response (to simulate the measured response) of the example systems, obtained by using TFM to integrate the nonlinear system equations with given system parameters. For computational ease, the impulse excitation was simulated by imposing initial velocity conditions. For all examples studied, all data points were used to determine the system parameters. However, for one example, results also were obtained using fewer (50) data points than those available (100 and 200). This (the use of two different meshes, a finer one for response generation and a coarser one for parameter identification) often is done in inverse problems to simulate the fact that the results from a finer mesh will be more representative (will contain higher-frequency components) of the physical system and using fewer data points for identification one can evaluate the robustness of the algorithm.

To simulate noise measurement, random noises with different rms (5, 10, and 20%) values were generated and added to the simulated response. The corresponding simulated corrupted data $\bar{u}_m$ are given by[19]

$$\bar{u}_m = u_m(1 + r) \tag{32}$$

where r is a uniformly distributed random number, generated by IMSL subroutine DRNUN, that is scaled and shifted to a range of $(-\alpha, \alpha)$ by using subroutines DSCAL and DADD. In this study, α was chosen as 0.05 and 0.1 for 10 and 20% noises, respectively. All computations were performed on an IBM 3090-300E mainframe computer.

System with Combined Quadratic and Cubic Nonlinearities

Consider a single-degree-of-freedom system[19]:

$$\ddot{u}(t) + a_1\dot{u}(t) + a_2u(t) + a_3u(t)^2 + a_4u(t)^3 = 0, \qquad 0 < t \leq 5 \tag{33}$$

$$u(0^+) = 0.0 \qquad \dot{u}(0^+) = 5.0 \tag{34}$$

for the following two cases of numerical values of the system parameters:

Case 1)

$$a_1 = 1.0, \quad a_2 = 25.0, \quad a_3 = 0.1, \quad a_4 = 0.5 \tag{35}$$

Case 2)

$$a_1 = 1.0, \quad a_2 = 25.0, \quad a_3 = 2.5, \quad a_4 = 5.0 \tag{36}$$

For case 1, initial values were chosen arbitrarily as $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 0.5$, $a_4 = 1.0$, and $a_5 = 3.5$; for case 2, the same were chosen as $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.0$, $a_4 = 2.5$, and $a_5 = 3.5$.

Tables 1 and 2 present the numerical results for both cases compared with those given by Normann and Kapania.[4] Record length was taken to be 5 s and was divided into 25 and 50 data points. In case 1, the identified parameter a_4 shows the worst accuracy when 25 data points of the 20% simulated corrupted data were used. The accuracy for a_4 improved by 70% when 50 data points were taken. The results for case (2) show a similar trend. For case 1, Figs. 1a and 1b present, respectively, the identified responses and the simulated data corrupted with 10 and 20% rms values. Figure 2 shows the sensitivities of the identified response with respect to system

Table 1 Numerical results for the system with combined quadratic and cubic nonlinearities (case 1)

Parameters	a_1	a_2	a_3	a_4	a_5
Initial values	0.7	22.5	0.5	1.0	3.5
			Data points = 25		
Identified (0%)[a]	1.00	25.00	0.10	0.50	5.00
Identified (10%)	1.01	24.87	0.25	0.90	5.04
Identified (20%)	1.03	24.74	0.41	1.29	5.08
			Data points = 50		
Identified (0%)	1.00	25.0	0.10	0.50	5.00
Identified (10%)	1.00	24.99	0.22	0.47	5.00
Identified (20%)	1.00	24.97	0.34	0.45	5.00
			Data points = 335[b]		
Identified (0%)	1.00	25.0	0.10	0.50	5.00
Identified (10%)	0.98	25.05	0.12	0.31	4.97
Identified (20%)	0.98	25.10	0.13	0.12	4.94
Exact values	1.0	25.0	0.1	0.5	5.0

[a]Noise levels in parentheses. [b]Results from Normann and Kapania.[4]

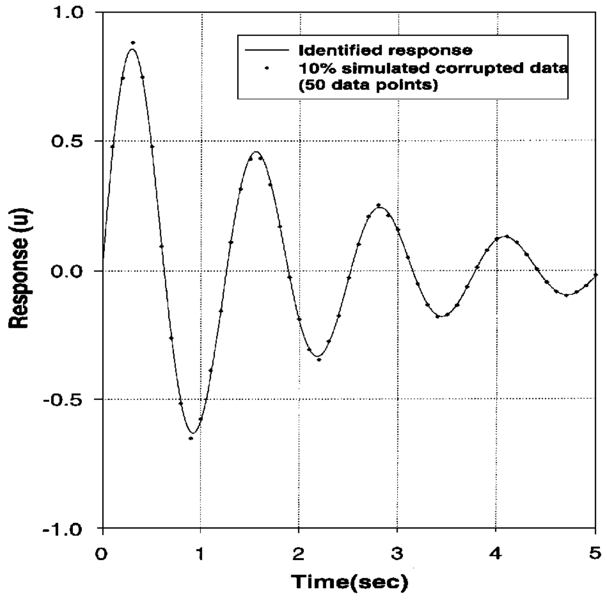

Fig. 1a Identified response of the system with combined quadratic and cubic nonlinearity using 10% simulated corrupted data (case 1).

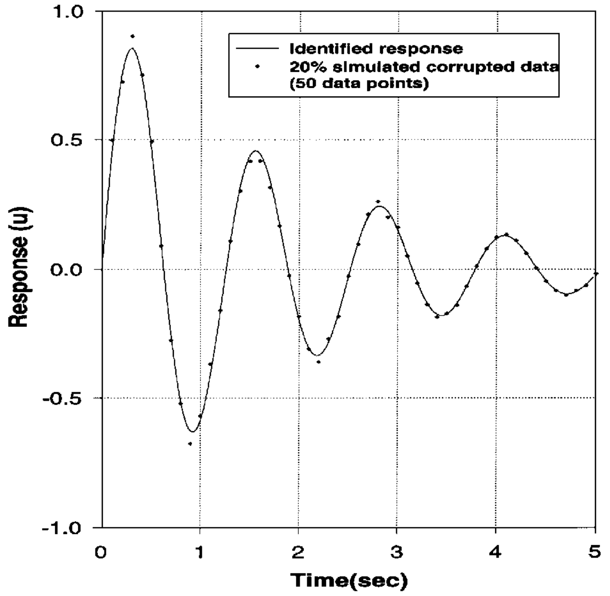

Fig. 1b Identified response of the system with combined quadratic and cubic nonlinearity using 20% simulated corrupted data (case 1).

Table 2 Numerical results for the system with combined quadratic and cubic nonlinearities (case 2)

Parameters	a_1	a_2	a_3	a_4	a_5
Initial values	0.7	22.5	1.0	2.5	3.5
		Data points $= 25$			
Identified (0%)[a]	1.00	25.00	2.50	5.00	5.00
Identified (10%)	0.99	24.99	2.32	5.28	4.91
Identified (20%)	0.97	24.96	2.11	5.61	4.81
		Data points $= 50$			
Identified (0%)	1.00	25.00	2.50	5.00	5.00
Identified (10%)	1.00	25.01	2.61	4.92	5.00
Identified (20%)	1.01	25.03	2.73	4.80	4.99
		Data points $= 335$[b]			
Identified (0%)	1.00	25.00	2.50	5.00	5.00
Identified (10%)	0.98	25.04	2.56	4.79	4.96
Identified (20%)	0.98	25.10	2.62	4.58	4.93
Exact values	1.0	25.0	2.5	5.0	5.0

[a]Noise levels in parentheses. [b]Results from Normann and Kapania.[4]

Table 3 Comparison of numerical results obtained from Levenberg–Marquardt and conjugate gradient methods for the system with combined quadratic and cubic nonlinearities, case 2, 50 data points

	Noise level, %								
p_k	0			10			20		
	50 data points selected from population sizes of								
	50	100	200	50	100	200	50	100	200
	Levenberg–Marquardt method								
a_1	1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.98	1.00
a_2	25.00	25.00	25.00	24.96	24.98	25.01	24.92	24.96	25.02
a_3	2.50	2.50	2.50	2.56	2.17	2.96	2.61	1.82	3.42
a_4	5.00	5.00	5.00	5.06	5.23	4.90	5.12	5.48	4.80
a_5	5.00	5.00	5.00	5.00	4.94	5.02	5.01	4.88	5.05
	Conjugate gradient method								
a_1	1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.98	1.01
a_2	25.00	25.00	25.00	24.95	24.97	25.00	24.92	24.95	25.01
a_3	2.50	2.53	2.53	2.60	2.19	3.01	2.67	1.82	3.47
a_4	5.00	5.28	5.28	5.34	5.53	5.16	5.34	5.79	5.06
a_5	5.00	5.01	5.01	5.01	4.95	5.03	5.01	4.89	5.05

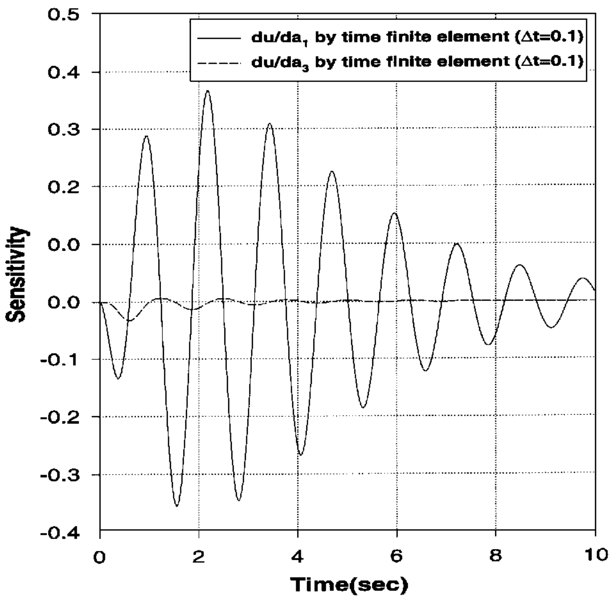

Fig. 2 Sensitivity of the identified response with respect to a_1 and a_3 for the system having combined quadratic and cubic nonlinearity (case 1).

parameters a_1 and a_3, respectively. It was observed that the parameter a_3 converges much slower than the other parameters. This is because the response is relatively insensitive to a_3 as compared to other parameters. Note that Normann and Kapania[4] used the record length as 5 s and 335 data points for representing the measured data.

For comparison purposes, the parameters for this example (case 2) also were identified using a conjugate gradient method. Table 3 presents results from this comparison using 50 data points. These 50 points were selected from population sizes of 50, 100, and 200 data points. The results show that Levenberg–Marquardt method is more accurate than the conjugate gradient method in this case. For noise-free case, the results were not influenced by the population size, but the results for 10 and 20% simulated corrupted data were. The inaccuracy of identified parameters for nonlinear terms was increased with an increase in the population size. Table 4 presents the results using 100 data points in the simulation. The 100 data points in the simulation were chosen from population sizes of 100 and 200 data points. The results show a trend similar to that of the 50-data-point case.

System with Linear and Quadratic Damping

Consider a single-degree-of-freedom system with linear and quadratic damping[4]:

$$\ddot{u}(t) + a_1\dot{u}(t) + a_2u(t) + a_3\dot{u}(t)\,|\dot{u}(t)| = 0, \quad 0 < t \le 5 \tag{37}$$

$$u(0^+) = 0.0 \qquad \dot{u}(0^+) = 5.0 \tag{38}$$

for the following two cases of system parameters:

Case 1)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 0.5 \tag{39}$$

Case 2)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 2.5 \tag{40}$$

As initial values, $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.0$, and $a_4 = 3.5$ were chosen for case 1, and $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.5$, and $a_4 = 3.5$ for case 2.

Tables 5 and 6 present the numerical results for both cases as compared to those given by Normann and Kapania.[4] For case 1, 25

Table 4 Comparison of numerical results obtained from Levenberg–Marquardt and conjugate gradient methods for the system with combined quadratic and cubic nonlinearities case 2, 100 data points

	Noise level, %					
p_k	0		10		20	
	100 data points selected from population sizes of					
	100	200	100	200	100	200
	Levenberg–Marquardt method					
a_1	1.00	1.00	0.99	0.99	0.99	0.99
a_2	25.00	25.00	25.00	24.99	25.01	24.99
a_3	2.50	2.50	2.18	2.89	1.85	3.28
a_4	5.00	5.00	5.03	4.92	5.07	4.84
a_5	5.00	5.00	4.97	4.99	4.94	4.98
	Conjugate gradient method					
a_1	1.00	1.00	0.99	0.99	0.99	0.99
a_2	25.00	25.00	25.00	24.99	25.01	24.99
a_3	2.51	2.50	2.18	2.90	1.85	3.30
a_4	5.07	5.07	5.10	4.98	5.15	4.90
a_5	5.00	5.00	4.97	4.99	4.94	4.98

Table 5 Numerical results for the system with quadratic damping (case 1)

Parameters	a_1	a_2	a_3	a_4
Initial values	0.7	22.5	0.5	3.5
		Data points $= 25$		
Identified (0%)[a]	1.00	25.00	0.50	5.00
Identified (10%)	0.99	25.04	0.51	5.04
Identified (20%)	0.97	25.09	0.53	5.08
		Data points $= 50$		
Identified (0%)	1.00	25.00	0.50	5.00
Identified (10%)	1.00	25.00	0.50	5.00
Identified (20%)	1.00	25.01	0.50	5.00
		Data points $= 335$[b]		
Identified (0%)	1.00	25.00	0.50	5.00
Identified (10%)	1.00	24.99	0.48	4.92
Identified (20%)	1.00	24.97	0.46	4.84
Exact values	1.00	25.00	0.50	5.0

[a]Noise levels in parentheses. [b]Results from Normann and Kapania.[4]

Table 6 Numerical results for the system with quadratic damping (case 2)

Parameters	a_1	a_2	a_3	a_4
Initial values	0.7	22.5	1.5	3.5
		Data points = 50		
Identified (0%)[a]	1.00	25.00	2.50	5.00
Identified (10%)	1.05	25.00	2.40	4.86
Identified (20%)	1.08	24.99	2.34	4.78
		Data points = 100		
Identified (0%)	1.00	25.00	2.50	5.00
Identified (10%)	1.00	24.97	2.49	5.00
Identified (20%)	0.99	24.94	2.52	5.04
		Data points = 335[b]		
Identified (0%)	1.00	25.00	2.50	5.00
Identified (10%)	0.96	24.96	2.49	4.92
Identified (20%)	0.94	24.92	2.49	4.84
Exact values	1.00	25.00	2.50	5.0

[a]Noise levels in parentheses. [b]Results from Normann and Kapania.[4]

and 50 data points were used; for case 2, 50 and 100 data points were used. Also we used integrated Legendre polynomials as basis functions which gave a better conditioned system of equations. For case 1, the present approach using 50 points gives results that are more accurate than those given by Normann and Kapania.[4] In case 2, the results using 100 data points show good accuracy. Figures 3a and 3b present, respectively, the identified responses and the simulated data corrupted with 10 and 20% noise, respectively, for case 2.

Two-Degree-of-Freedom System Having Cubic Nonlinearities

Consider a two-degree-of-freedom system having cubic nonlinearities,[20]

$$\ddot{u}_1 = -\omega_1^2 u_1 - 2\mu_1 \dot{u}_1 - \alpha_1 u_1^3 - \alpha_2 u_1^2 u_2 - \alpha_3 u_1 u_2^2 - \alpha_4 u_2^3 \tag{41a}$$

$$\ddot{u}_2 = -\omega_2^2 u_2 - 2\mu_2 \dot{u}_2 - \alpha_5 u_1^3 - \alpha_6 u_1^2 u_2 - \alpha_7 u_1 u_2^2 - \alpha_8 u_2^3 \tag{41b}$$

with initial conditions

$$u_1(0) = 1.5, \quad \dot{u}_1(0) = 0.0, \quad u_2(0) = -1.0, \quad \dot{u}_2(0) = 0.0 \tag{42}$$

and the following values of system parameters:

$$\omega_1^2 = 25.0, \quad \mu_1 = 0.35, \quad \alpha_1 = 5.0, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0.25, \quad \alpha_4 = 3.0 \tag{43a}$$

$$\omega_2^2 = 17.0, \quad \mu_2 = 0.25, \quad \alpha_5 = 2.5, \quad \alpha_6 = 0.75, \quad \alpha_7 = 0.2, \quad \alpha_8 = 5.0 \tag{43b}$$

This type of systems is associated with many physical systems such as the vibration of strings, beams, and plates.

One-Step Identification Procedure

The domain ($0 < t \leq 10$) is divided into 100 elements of equal time steps. Legendre polynomials of the second degree are used as basis functions for the TFM. For simplicity, initial velocities have not been treated as unknown parameters. The parameters to be identified are $\{\omega_1^2, \mu_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \omega_2^2, \mu_2, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. The initial trial values of system parameters were chosen as

$$\omega_1^2 = 20.0, \quad \mu_1 = 0.50, \quad \alpha_1 = 3.0, \quad \alpha_2 = 0.20, \quad \alpha_3 = 0.15, \quad \alpha_4 = 4.0 \tag{44a}$$

$$\omega_2^2 = 14.0, \quad \mu_2 = 0.55, \quad \alpha_5 = 4.5, \quad \alpha_6 = 0.35, \quad \alpha_7 = 0.35, \quad \alpha_8 = 2.0 \tag{44b}$$

The results (Table 7) show that the method after 1000 iterations did not converge to any values, the parameters for nonlinear terms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ were particularly unreasonable.

Table 7 Numerical results for the two-degree-of-freedom system using one-step procedure and no noise

p_k	Initial	Identified, 0% noise	Exact
ω_1^2	20.0	26.07	25.00
μ_1	0.5	0.48	0.35
α_1	3.0	3.60	5.00
α_2	0.2	2.32	0.50
α_3	0.15	1.54	0.25
α_4	4.0	−0.96	3.00
ω_2^2	14.0	18.38	17.00
μ_2	0.55	0.31	0.25
α_5	4.5	1.85	2.50
α_6	0.35	−4.08	0.75
α_7	0.35	−3.95	0.20
α_8	2.0	3.63	5.00

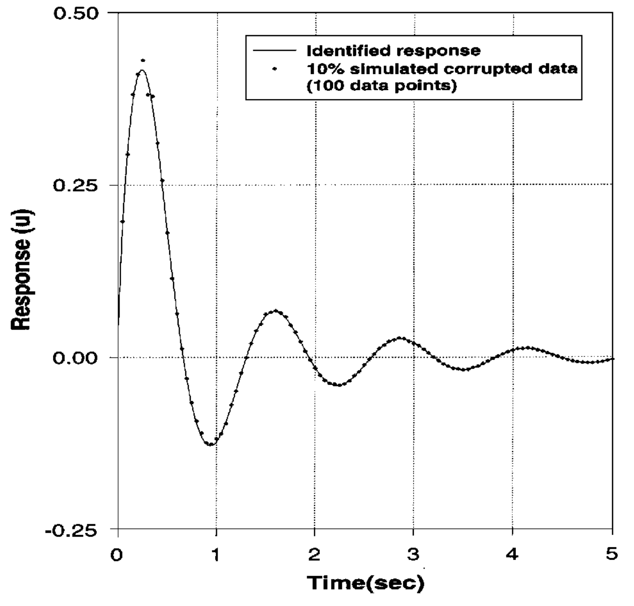

Fig. 3a Identified response of the system with linear and quadratic damping using 10% simulated corrupted data (case 2).

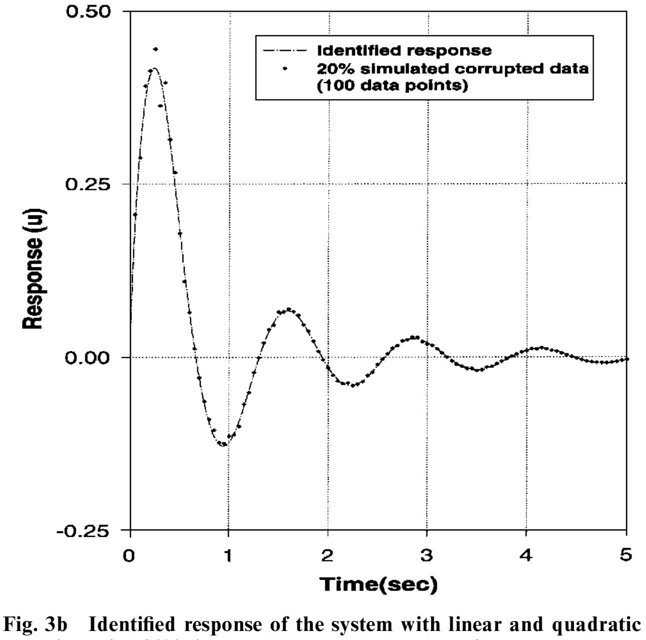

Fig. 3b Identified response of the system with linear and quadratic damping using 20% simulated corrupted data (case 2).

Table 8 Numerical results for the two-degree-of-freedom system using two-step procedure

p_k	Initial	Noise, %				Exact
		0	5	10	20	
ω_1^2	20.0	25.00 (25.00)[a]	25.00 (24.58)	25.01 (24.23)	25.01 (23.72)	25.00
μ_1	0.50	0.35 (0.35)	0.35 (0.34)	0.35 (0.34)	0.35 (0.35)	0.35
α_1	3.0	5.00 (5.00)	5.24 (5.97)	5.49 (6.76)	6.01 (7.77)	5.00
α_2	0.20	0.49 (0.50)	0.86 (1.72)	1.25 (2.72)	2.04 (4.07)	0.50
α_3	0.15	0.25 (0.25)	0.10 (0.77)	−0.07 (1.23)	−0.42 (1.99)	0.25
α_4	4.00	3.00 (3.00)	2.85 (2.92)	2.68 (2.79)	2.35 (2.44)	3.00
ω_2^2	14.00	17.00 (17.00)	17.00 (17.11)	17.00 (17.27)	16.99 (17.75)	17.00
μ_2	0.55	0.25 (0.25)	0.25 (0.26)	0.25 (0.26)	0.25 (0.27)	0.25
α_5	4.50	2.50 (2.50)	2.53 (2.73)	2.56 (2.93)	2.64 (3.21)	2.50
α_6	0.35	0.75 (0.75)	0.91 (0.61)	1.10 (0.34)	1.48 (−0.56)	0.75
α_7	0.35	0.20 (0.20)	0.32 (−0.30)	0.45 (−0.91)	0.69 (−2.40)	0.20
α_8	2.00	5.00 (5.00)	4.95 (4.75)	4.89 (4.40)	4.76 (3.46)	5.00

[a]Parentheses indicate results from Normann and Kapania[4] using 335 data points, unfiltered data, and the same initial guess values.

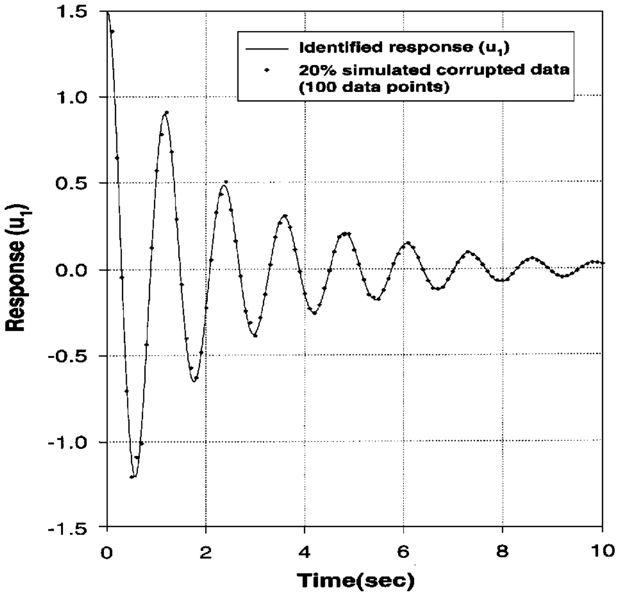

Fig. 4a Identified response u_1 of two-degree-of-freedom system having cubic nonlinearities using 20% simulated corrupted data.

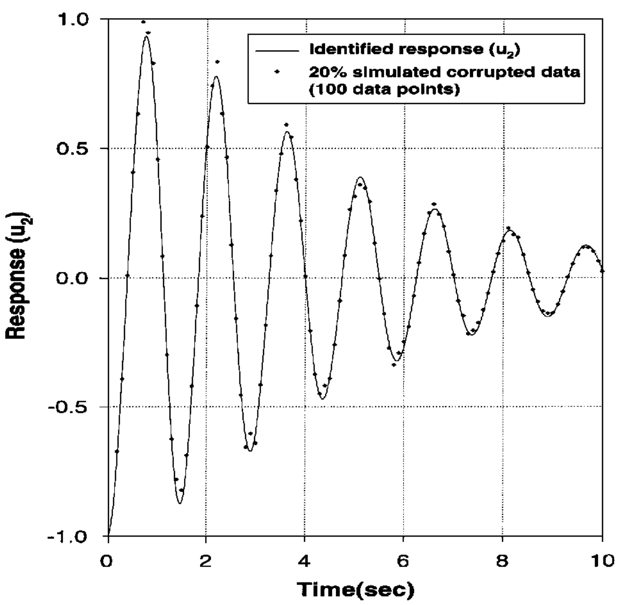

Fig. 4b Identified response u_2 of two-degree-of-freedom system having cubic nonlinearities using 20% simulated corrupted data.

Two-Step Identification Procedure

A two-step procedure was then adopted for the direct iterative method. First, the parameters for the linear terms, $\{\omega_1^2, \mu_1, \omega_2^2, \mu_2\}$, were identified, then those for the nonlinear terms. For linear terms, the simulated data were generated using very small initial displacements $[u_1(0) = 0.05, u_2(0) = -0.03]$. Because the original system, Eqs. (41a) and (41b), behaves almost linearly when subjected to very small initial conditions, identification was performed assuming the system to be linear:

$$\ddot{u}_1 = -\omega_1^2 u_1 - 2\mu_1 \dot{u}_1; \qquad u_1(0) = 0.05, \qquad \dot{u}_1(0) = 0.0 \tag{45a}$$

$$\ddot{u}_2 = -\omega_2^2 u_2 - 2\mu_2 \dot{u}_2; \qquad u_2(0) = -0.03, \qquad \dot{u}_2(0) = 0.0 \tag{45b}$$

The simulated data, however, were obtained by integrating the actual nonlinear system. Next, the parameters for nonlinear terms were identified by keeping the parameters corresponding to linear terms $\{\omega_1^2, \mu_1, \omega_2^2, \mu_2\}$ as fixed. Identification was performed on the following system:

$$\ddot{u}_1 = -\omega_1^2 u_1 - 2\mu_1 \dot{u}_1 - \alpha_1 u_1^3 - \alpha_2 u_1^2 u_2 - \alpha_3 u_1 u_2^2 - \alpha_4 u_2^3 \tag{46a}$$

$$\ddot{u}_2 = -\omega_2^2 u_2 - 2\mu_2 \dot{u}_2 - \alpha_5 u_1^3 - \alpha_6 u_1^2 u_2 - \alpha_7 u_1 u_2^2 - \alpha_8 u_2^3 \tag{46b}$$

$$u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0 \tag{47}$$

Table 8 presents the numerical results of the identified system using 100 points of the simulated data. The results using noise-free simulated data easily converged to the given values of system parameters. Also, results using data corrupted by 5, 10, and 20% random noise show reasonably accurate values of system parameters. Figures 4a and 4b present, respectively, the identified responses for u_1 and u_2 with the simulated data corrupted by 20% random noise.

Conclusions

The time-finite-element-based iterative direct method is found to be very effective for the identification of parameters in the nonlinear examples studied here. For the two-degree-of-freedom nonlinear system, a two-step approach in which parameters corresponding to linear and nonlinear terms are identified separately was necessary for convergence. In comparison with the results of Normann and Kapania, the proposed method uses fewer data points to obtain the same accuracy. Moreover, by using a time finite element formulation, not only the transient responses but also sensitivities of the transient response are calculated easily (by performing a direct differentiation of the resulting algebraic equations). Good results were obtained for the presented examples. Based on the results presented here, the proposed method appears to be a good choice for performing parametric identification of nonlinear systems.

References

[1]Natke, H. G., Juang, J. N., and Gawronski, W., "A Brief Review on the Identification of Nonlinear Mechanical Systems," *Proceedings of the 6th International Modal Analysis Conference* (Kissimmee, FL), Vol. 2, Society for Experimental Mechanics, Bethel, CT, 1988, pp. 1569–1574.

[2]Batill, S. M., and Bacarro, J. M., "Modeling and Identification of Nonlinear Dynamic Systems with Application to Aircraft Landing Gear," AIAA Paper 88-2315, April 1988.

[3]Mook, D. J., "Estimation and Identification of Nonlinear Dynamic Systems," *AIAA Journal*, Vol. 27, No. 7, 1989, pp. 968–974.

[4]Normann, J. B., and Kapania, R. K., "Parametric Identification of Nonlinear Structural Dynamic Systems," AIAA Paper 90-1232, April 1990.

[5]Hamel, P. G., and Jategaonkar, R. V., "Evolution of Flight Vehicle System Identification," *Journal of Aircraft*, Vol. 33, No. 1, 1996, pp. 9–28.

[6]Argyris, J. H., and Scharpf, D. W., "Finite Element in Time and Space," *Journal of the Royal Aeronautical Society*, Vol. 73, No. 708, 1969, pp. 1041–1044.

[7]Bailey, C. D., "A New Look at Hamilton's Principle," *Foundations of Physics*, Vol. 5, No. 3, 1975, pp. 433–451.

[8]Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 13, No. 9, 1975, pp. 1154–1157.

[9]Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, No. 2, 1976, pp. 235–247.

[10]Bailey, C. D., "Hamilton, Ritz, and Elastodynamics," *Journal of Applied Mechanics*, Vol. 43, No. 4, 1976, pp. 684–688.

[11]Peters, D. A., and Izadpanah, A. P., "hp-Version Finite Elements for the Space-Time Domain," *Computational Mechanics*, Vol. 3, No. 2, 1988, pp. 73–88.

[12]Kapania, R. K., and Park, S., "Nonlinear Transient Response and Its Sensitivity Using Finite Elements in Time," *Computational Mechanics*, Vol. 17, No. 5, 1996, pp. 306–317.

[13]Hanagud, S. V., Meyyappa, M., and Craig, J. I., "Method of Multiple Scales and Identification of Nonlinear Structural Dynamic Systems," *AIAA Journal*, Vol. 23, No. 5, 1985, pp. 802–807.

[14]Levenberg, K., "A Method for the Solution of Certain Nonlinear Problems in Least Squares," *Quarterly of Applied Mathematics*, Vol. 2, No. 2, 1944, pp. 164–168.

[15]Marquardt, D. W., "An Algorithm for Least-Squares Estimation of Nonlinear Parameters," *Journal of the Society for Industrial and Applied Mathematics*, Vol. 11, No. 2, 1963, pp. 431–441.

[16]Kapania, R. K., and Normann, J. B., "Parametric Identification of Nonlinear Multiple Degree-of-Freedom Structural Dynamic Systems," *Proceedings of the 4th International Conference on Structural Dynamics: Recent Advances*, Univ. of Southampton, England, UK, 1991, pp. 411–421.

[17]Fletcher, R., "Conjugate Direction Methods," *Practical Methods of Optimization*, 2nd ed., Wiley, New York, 1987, pp. 80–92.

[18]Fletcher, R., and Reeves, C. M., "Function Minimization by Conjugate Gradients," *Computer Journal*, Vol. 7, No. 2, 1964, pp. 149–154.

[19]Distefano, N., and Rath, A., "System Identification in Nonlinear Structural Seismic Dynamics," *Computer Methods in Applied Mechanics and Engineering*, Vol. 5, 1975, pp. 353–372.

[20]Nayfeh, A. H., and Mook, D. T., "Free Oscillations of Systems Having Cubic Nonlinearities," *Nonlinear Oscillations*, Wiley, New York, 1979, pp. 387–395.

A. Berman
Associate Editor